Last time:

wave equation for membrane

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

$$u\big|_{\partial M} = 0 \qquad \text{BC}$$

$$u(x,y,0) = \alpha(x,y)$$

$$\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$$

considered product solutions

$$u(x,y,t) = \phi(x,y)\, h(t)$$

$\Rightarrow$ ODE: $h''(t) = -c^2 \lambda h(t)$

PDE: $$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi}$$

rectangular membrane

$\Rightarrow$ can write $\boxed{\phi(x,y) = f(x)\, g(y)}$

BC:

$u(0,y,t) = 0 \Rightarrow f(0) = 0$

$u(L,y,t) = 0 \Rightarrow f(L) = 0$

bottom: $u(x,0,t) = 0 \Rightarrow g(0) = 0$

$u(x,H,t) = 0 \Rightarrow g(H) = 0$

PDE $\rightarrow$ $\boxed{f''(x)\, g(y) + f(x)\, g''(y) = -\lambda f(x)\, g(y)}$

$1/f(x)g(y)$

get $$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} = -\lambda$$

$$\frac{f''(x)}{f(x)} = \boxed{-\lambda - \frac{g''(y)}{g(y)} = -\mu}$$

$\uparrow$ only depends on x

$\uparrow$ only depends on y

$\Rightarrow$ get $$\frac{f''(x)}{f(x)} = -\mu$$

$\Leftrightarrow$
$$\boxed{\begin{aligned} f''(x) &= -\mu f(x) \\ f(0) &= 0 = f(L) \end{aligned}}$$

as before:

possible eigenvalues are

$$\boxed{\mu = \frac{n^2\pi^2}{L^2}, \quad n=1,2,3,\dots \\ \text{with eigenfunction} \quad \sin \frac{n\pi}{L} x}$$

yellow equation

$$\frac{g''(y)}{g(y)} = -\lambda + \mu = -(\lambda - \mu)$$

$$\Rightarrow \quad \boxed{\begin{aligned} g''(y) &= -(\lambda-\mu)\, g(y) \\ g(0) &= 0 = g(H) \end{aligned}}$$

$\lambda - \mu$ eigenvalue

$$\Rightarrow \lambda - \mu = \frac{m^2\pi^2}{H^2}$$

$m=1,2,3,$

$$\boxed{\lambda - \mu = \frac{m^2\pi^2}{H^2}} \qquad \left(\mu = \frac{n^2\pi^2}{L^2}\right)$$

$$\Rightarrow \boxed{\lambda = \lambda_{nm} = \frac{m^2\pi^2}{H^2} + \frac{n^2\pi^2}{L^2}, \quad \begin{matrix} n = 1, 2, 3 \ldots \\ m = 1, 2, 3 \ldots \end{matrix}}$$

we had $g''(y) = (\lambda - \mu) g(y)$, $g(0) = 0 = g(H)$

$$\Rightarrow \boxed{g(y) = \sin \frac{m\pi}{H} y}$$

<u>**Result**</u>: $\nabla^2 (f(x) g(y)) = -\lambda f(x) g(y)$ with given bd. conditions

has eigenvalues $\lambda_{nm} = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}$

with eigenfunctions $f(x) g(y) = \sin \frac{n\pi}{L} x \, \sin \frac{m\pi}{H} y$

$\overset{\shortparallel}{\phi(x,y)}$

Determine $h(t)$:

$$h''(t) = -\lambda c^2 h(t)$$
$$= -\underbrace{\lambda_{nm} c^2}_{\text{positive}} h(t)$$

$$\Rightarrow h(t) = A_{nm} \cos c\sqrt{\lambda_{nm}}\, t + B_{nm} \sin c\sqrt{\lambda_{nm}}\, t$$

$\Rightarrow$ General solution

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \cos c\sqrt{\lambda_{nm}}\, t$$
$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \quad " \quad " \quad \sin c\sqrt{\lambda_{nm}}\, t$$

where $\lambda_{nm} = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}$

To determine coefficients A_{nm} and B_{nm} we use initial conditions

$$u(x,y,0) = \alpha(x,y)$$

$$\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$$

$$\Rightarrow \quad \beta(x,y) = \frac{\partial u}{\partial t}(x,y,0) =$$

$$= \sum_{n,m} B_{nm} \sin\frac{n\pi}{L}x \, \sin\frac{m\pi}{H}y \; c\sqrt{\lambda_{nm}} \; \cos c\sqrt{\lambda_{nm}}\, t \qquad \uparrow\ t=0$$

$$+ \sum_{n,m} A_{nm} \quad '' \quad '' \quad '' \quad \sin c\sqrt{\cdot}\; t \qquad \uparrow\ t=0 \;\; = 0$$

$$\boxed{\beta(x,y) = \sum_{n,m} \underline{B_{nm}\, c\sqrt{\lambda_{nm}}} \; \underline{\sin\frac{n\pi}{L}x \, \sin\frac{m\pi}{H}y}}$$

Get coefficient B_{nm} via double integral

$$\boxed{c\sqrt{\lambda_{nm}}\, B_{nm} = \frac{2}{L}\frac{2}{H}\int_0^L\int_0^H \beta(x,y)\sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{H}y\,dy\,dx}$$

Exercise:
write down formula for A_{nm}, using $u(x,y,0) = \alpha(x,y)$

Remark about review problem:
separate variables
$\Rightarrow$ get ODE $h''(t) + \beta h'(t) + \gamma h(t) = 0$
consider polynomial $x^2 + \beta x + \gamma = 0$
has solution $x_{1,2} = \frac{1}{2}\left(-\beta \pm \sqrt{\beta^2 - 4\gamma}\right)$
if $\beta^2 - 4\gamma < 0$ $\Rightarrow$ ODE has general solution
$h(t) = e^{-\beta t/2}\left(A\cos\sqrt{4\gamma-\beta^2}\,t + B\sin\sqrt{4\gamma-\beta^2}\,t\right)$

For 4.4.3(b)